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The Bosonic Vertex Operator Algebra on a Genus g Riemann Surface

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Abstract
We discuss the partition function for the Heisenberg vertex operator algebra on a genus g Riemann surface formed by sewing g handles to a Riemann sphere. In particular, it is shown how the partition can be computed by means of the MacMahon Master Theorem from classical combinatorics.

1 Introduction

In this paper we briefly sketch recent progress in defining and computing the partition function for the Heisenberg Vertex Operator Algebra (VOA) on a genus g Riemann surface. The partition function and n-point correlation functions are familiar concepts at genus one and have recently been computed on genus two Riemann surfaces formed from sewing tori together [MT1],[MT2]. Here we discuss an alternative approach for computing these objects on a general genus g Riemann surface formed by sewing g handles onto a Riemann sphere. This approach includes the classical Schottky parameterisation and a related simpler canonical parameterisation for which we obtain the partition function for rank 2 Heisenberg VOA in terms of an explicit infinite determinant. This determinant is computed by means of the MacMahon Master Theorem in classical combinatorics [MM].

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2 A Generalized MacMahon Master Theorem

We begin with a review of the MacMahon Master Theorem and a recent generalization. We will provide a proof of this which gives some flavour of the combinatorial graph theory methods developed to compute higher genus partition functions [MT2], [TZ].

Let $A = (A_{ij})$ be an $n \times n$ matrix indexed by $i, j \in \{1, \ldots, n\}$. Consider the cycle decomposition of $\pi \in \Sigma_n$, the symmetric group on $\{1, \ldots, n\}$,

$$\pi = \sigma_1 \ldots \sigma_{C(\pi)}. \tag{1}$$

The β-extended Permanent of the matrix A is defined by [FZ]

$$\mathrm{perm}_\beta A = \sum_{\pi \in \Sigma_n} \beta^{C(\pi)} \prod_i A_{i\pi(i)}. \tag{2}$$

The standard permanent and determinant are the particular cases:

$$\mathrm{perm}\, A = \mathrm{perm}_{+1} A, \qquad \det A = (-1)^n \mathrm{perm}_{-1} A. \tag{3}$$

Consider a multiset $\{k_1, \ldots, k_m\}$ with $1 \le k_1 \le \ldots \le k_m \le n$ i.e. index repetition is allowed. We notate the multiset as the unrestricted partition

$$\mathbf{k} = \{1^{r_1} 2^{r_2} \ldots n^{r_n}\}, \tag{4}$$

i.e. the index i occurs $r_i \ge 0$ times and where $m = \sum_{i=1}^{n} r_i$. Let $A(\mathbf{k})$ denote the $m \times m$ matrix indexed by $\mathbf{k}$ for a given matrix A indexed by $\{1, \ldots, n\}$. We now describe a generalisation of the classic MacMahon Master Theorem (MMT) of combinatorics [MM]. Let A be an $n \times n$ matrix indexed by $\{1, \ldots, n\}$. Let $A(\mathbf{k})$ denote the $m \times m$ matrix indexed by a multiset $\mathbf{k}$ (4).

Theorem 2.1 (Generalized MMT - Foata and Zeilberger [FZ])

$$\sum_{\mathbf{k}} \frac{\mathrm{perm}_\beta A(\mathbf{k})}{r_1! r_2! \ldots r_n!} = \frac{1}{\det(I - A)^\beta}, \tag{5}$$

where the (infinite) sum ranges over all multisets $\mathbf{k} = \{1^{r_1}.2^{r_2} \ldots .n^{r_n}\}$.

For $\beta = 1$, Theorem 2.1 reduces to the classical MMT [MM]. For $\beta = -1$ we use (3) to find that the sum is restricted to proper subsets of $\{1, 2, \ldots, n\}$ resulting in the determinant identity

$$\det(I + B) = \sum_{1 \le k_1 < \ldots < k_m \le n} \det B(\mathbf{k}),$$

for $B = -A$.

Proof of Theorem 2.1. We use a graph theory method applied in [MT2]. Define a set of oriented graphs Γ with elements γ_π whose vertices are labelled by multisets $\mathbf{k} = \{1^{r_1} \ldots n^{r_n}\}$ and directed edges $\{e_{ij}\}$ determined by permutations $\pi \in \Sigma(\mathbf{k})$ as follows

$$e_{ij} = \overset{k_i}{\bullet} \longrightarrow \overset{k_j}{\bullet} \quad \text{for } k_j = \pi(k_i)$$

Define a β dependent weight for each γ_π

$$w_\beta(e_{ij}) = A_{k_i k_j}, \qquad w_\beta(\gamma_\pi) = \beta^{C(\pi)} \prod\nolimits_{e_{ij} \in \gamma_\pi} w_\beta(e_{ij}), \tag{6}$$

where $C(\pi)$ is the number of disjoint cycles in π. Then we may write

$$\mathrm{perm}_\beta\, A(\mathbf{k}) = \sum_{\pi \in \Sigma(\mathbf{k})} w_\beta(\gamma_\pi).$$

γ_π is invariant under permutations of the identical labels of $\mathbf{k}$. Hence the left hand side of (5) can be rewritten as

$$\sum_{\mathbf{k}} \frac{\mathrm{perm}_\beta\, A(\mathbf{k})}{r_1! r_2! \ldots r_n!} = \sum_{\gamma \in \Gamma} \frac{w_\beta(\gamma)}{|\mathrm{Aut}(\gamma)|},$$

where we sum over all inequivalent graphs in Γ. Each $\gamma \in \Gamma$ can be decomposed into disjoint connected cycle graphs $\gamma_\sigma \in \Gamma$

$$\gamma = \gamma_{\sigma_1}^{m_1} \cdots \gamma_{\sigma_K}^{m_K}.$$

Each cycle σ corresponds to a disjoint connected cycle graph $\gamma_\sigma \in \Gamma$ with weight

$$w_\beta(\gamma_\pi) = \prod_i w_\beta(\gamma_{\sigma_i})^{m_i}.$$

Furthermore

$$|\mathrm{Aut}(\gamma_\pi)| = \prod_i |\mathrm{Aut}(\gamma_{\sigma_i})|^{m_i} m_i!$$

Let Γ_σ denote the set of inequivalent cycles. Then

$$\begin{aligned}\sum_{g\in\Gamma} \frac{w_\beta(g)}{|\mathrm{Aut}(g)|} &= \prod_{\gamma_\sigma\in\Gamma_\sigma} \sum_{m\geq 0} \frac{w_\beta(\gamma_\sigma)^m}{|\mathrm{Aut}(\gamma_\sigma)|^m m!} \\ &= \exp\left(\sum_{\gamma_\sigma\in\Gamma_\sigma} \frac{w_\beta(\gamma_\sigma)}{|\mathrm{Aut}(\gamma_\sigma)|}\right). \qquad (7)\end{aligned}$$

For a cycle σ of order $|\sigma| = r$ then $\mathrm{Aut}(\gamma_\sigma) = \langle \sigma^s \rangle$, a cyclic group of order $|\mathrm{Aut}(\gamma_\sigma)| = \frac{r}{s}$. Using the trace identity

$$\sum_{\gamma_\sigma, |\sigma|=r} s\, w_\beta(\gamma_\sigma) = \beta \mathrm{Tr}(A^r),$$

we find

$$\begin{aligned}\sum_{\gamma_\sigma\in\Gamma_\sigma} \frac{w_\beta(\gamma_\sigma)}{|\mathrm{Aut}(\gamma_\sigma)|} &= \beta \sum_{r\geq 1} \frac{1}{r} \mathrm{Tr}(A^r) \\ &= -\beta \mathrm{Tr}(\log(I - A)) \\ &= -\beta \log\det(I - A).\end{aligned}$$

Thus

$$\sum_{\mathbf{k}} \frac{\mathrm{perm}_\beta\, A(\mathbf{k})}{r_1! r_2! \ldots r_n!} = \det(I - A)^{-\beta}. \quad \square$$

Define a cycle to be primitive (or rotationless) if $|\mathrm{Aut}(\gamma_\sigma)| = 1$. For a general cycle σ with $|\mathrm{Aut}(\gamma_\sigma)| = s$ we have for $\beta = 1$

$$w_1(\gamma_\sigma) = w_1(\gamma_\rho)^s,$$

for some primitive cycle ρ. Let Γ_ρ denote the set of all primitive cycles. Then

$$\begin{aligned}\sum_{\gamma_\sigma\in\Gamma_\sigma} \frac{w_1(\gamma_\sigma)}{|\mathrm{Aut}(\gamma_\sigma)|} &= \sum_{\gamma_\rho\in\Gamma_\rho} \sum_{s\geq 1} \frac{1}{s} w_1(\gamma_\rho)^s \\ &= -\sum_{\gamma_\rho\in\Gamma_\rho} \log\det(1 - w_1(\gamma_\rho)).\end{aligned}$$

Combining this with (7) implies [MT2]

Theorem 2.2

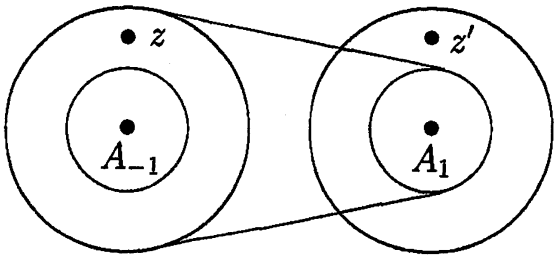
$$\det(I - A) = \prod_{\gamma_\rho \in \Gamma_\rho} (1 - w_1(\gamma_\rho)).$$

3 Riemann Surfaces from a Sewn Sphere

3.1 The Riemann torus

Consider the construction of a torus by sewing a handle to the Riemann sphere $\hat{\mathbb{C}}$ by identifying annular regions centred at $A_{\pm 1} \in \hat{\mathbb{C}}$ via a sewing condition with complex sewing parameter ρ

$$(z - A_{-1})(z' - A_1) = \rho. \tag{8}$$



We call $\rho, A_\pm$ canonical parameters. The annuli do not intersect provided

$$|\rho| < \frac{1}{4}|A_{-1} - A_1|^2. \tag{9}$$

Inequivalent tori depend only on

$$\chi = -\frac{\rho}{(A_{-1} - A_1)^2}, \tag{10}$$

where (9) implies $|\chi| < \frac{1}{4}$ [MT1].

Equivalently, we define $q, a_{\pm 1}$, known as Schottky parameters, by

$$\begin{aligned} a_i &= \frac{A_i + qA_{-i}}{1+q}, \\ \frac{q}{(1+q)^2} &= \chi, \end{aligned} \tag{11}$$

for $i = \pm 1$. Inequivalent tori depend only on q with $|q| < 1$. The canonical sewing condition (8) is equivalent to:

$$\left(\frac{z - a_{-1}}{z - a_1}\right)\left(\frac{z' - a_1}{z' - a_{-1}}\right) = q. \tag{12}$$

Inverting (11) we find that $q = C(\chi)$ for Catalan series

$$C(\chi) = \frac{1 - (1 - 4\chi)^{1/2}}{2\chi} - 1 = \sum_{n \geq 1} \frac{1}{n} \binom{2n}{n+1} \chi^n. \tag{13}$$

3.2 Genus g Riemann Surfaces

We may similarly construct a general genus g Riemann surface by identifying g pairs of annuli centred at $A_{\pm i} \in \hat{\mathbb{C}}$ for $i = 1, \ldots, g$ and sewing parameters ρ_i satisfying

$$(z - A_{-i})(z' - A_i) = \rho_i, \tag{14}$$

provided no two annuli intersect. Equivalently, for $i = 1, \ldots, g$ we define Schottky parameters $a_{\pm i}, q_i$ by

$$\begin{aligned} a_{\pm i} &= \frac{A_{\pm i} + q A_{\mp i}}{1 + q_i}, \\ \frac{q_i}{(1 + q_i)^2} &= -\frac{\rho_i}{(A_{-i} - A_i)^2}, \end{aligned} \tag{15}$$

where $|q_i| < 1$ is again related to the Catalan series (13)

$$q_i = C(\chi_i), \quad \chi_i = -\frac{\rho_i}{(A_i - A_{-i})^2}.$$

The canonical sewing condition can then be rewritten as a standard Schottky sewing condition:

$$\left(\frac{z - a_{-i}}{z - a_i}\right)\left(\frac{z' - a_i}{z' - a_{-i}}\right) = q_i. \tag{16}$$

The Schottky sewing condition (16) determines a Möbius map $z' = \gamma_i(z)$ where

$$\gamma_i = \sigma_i^{-1} \begin{pmatrix} q_i & 0 \\ 0 & 1 \end{pmatrix} \sigma_i, \tag{17}$$

for Möbius map

$$\sigma_i(z) = \frac{z - a_i}{z - a_{-i}}. \tag{18}$$

We define the Schottky group $\Gamma = \langle \gamma_i \rangle$ as the Kleinian group freely generated by γ_i for $i = 1, \ldots, g$.

One can find explicit formulas for various objects defined on the Riemann surface such as the bilinear form of the second kind, a basis of g holomorphic 1-forms and the genus g period matrix in terms of either the Canonical or Schottky parametrizations [TZ]. In the Schottky case, these involve sums or products over the Schottky group or subsets thereof.

4 Vertex Operator Algebras

Consider a simple VOA with $\mathbb{Z}$-graded vector space $V = \oplus_{n \geq 0} V^{(n)}$ and local vertex operators $Y(a,z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$ for $a \in V$ e.g. [Ka],[FLM],[MN],[MT3]. We assume that V is of CFT type (i.e. $V_0 = \mathbb{C}\mathbf{1}$) with a unique symmetric invertible invariant bilinear form $\langle\ ,\ \rangle$ with normalization $\langle \mathbf{1}, \mathbf{1} \rangle = 1$ where [FHL],[Li]

$$\langle Y(a,z)b, c \rangle = \langle b, Y(e^{zL_1}(-\frac{1}{z^2})^{L_0} a, \frac{1}{z})c \rangle \tag{19}$$

For a V-basis $\{u^\alpha\}$, we let $\{\overline{u}^\alpha\}$ denote the dual basis. If $a \in V^{(k)}$ is quasi-primary ($L_1 a = 0$) then (19) implies

$$\langle a_n b, c \rangle = (-1)^k \langle b, a_{2k-n-2} c \rangle.$$

In particular:

$$\begin{aligned} \langle a_n b, c \rangle &= -\langle b, a_{-n} c \rangle \text{ for } a \in V^{(1)} \\ \langle L_n b, c \rangle &= \langle b, L_{-n} c \rangle \text{ for } \omega \in V^{(2)}, \end{aligned} \tag{20}$$

so that b, c with unequal weights are orthogonal.

4.1 Genus Zero Correlation Functions

For $u_1, u_2, \ldots, u_n \in V$ define the n-point (correlation) function by

$$\langle \mathbf{1}, Y(u_1, z_1) Y(u_2, z_2) \ldots Y(u_n, z_n) \mathbf{1} \rangle. \tag{21}$$

The locality property of vertex operators implies that this formal expression (21) coincides with the analytic expansion of a rational function of $z_1, z_2, \ldots, z_n$ in the domain $|z_1| > |z_2| > \ldots > |z_n|$. Thus the n-point function can taken to be a rational function of $z_1, z_2, \ldots, z_n \in \hat{\mathbb{C}}$, the Riemann sphere in the domain. For example [HT]

Theorem 4.1 *For a VOA of central charge C, the Virasoro n-point function is a β-extended permanent*

$$\langle \mathbf{1}, Y(\omega, z_1) \ldots Y(\omega, z_n)\mathbf{1}\rangle = \mathrm{perm}_{\frac{c}{2}} B,$$

for $B_{ij} = \frac{1}{(z_i - z_j)^2}, i \neq j$ and $B_{ii} = 0$.

4.2 Rank Two Heisenberg VOA M_2

Consider the VOA generated by two Heisenberg vectors $a^{\pm} \in V^{(1)}$ whose modes satisfy non-trivial commutator

$$[a_m^+, a_n^-] = m\delta_{m,-n}. \tag{22}$$

V has a Fock basis spanned by

$$a_{\mathbf{k},\mathbf{l}} = a_{-k_1}^+ \ldots a_{-k_m}^+ a_{-l_1}^- \ldots a_{-l_n}^- \mathbf{1}, \tag{23}$$

labelled by a multisets $\mathbf{k} = \{k_1, \ldots, k_m\} = \{1^{r_1}.2^{r_2}\ldots\}$ and $\mathbf{l} = \{l_1, \ldots, l_n\} = \{1^{s_1}.2^{s_2}\ldots\}$. The Fock vectors are orthogonal with respect to to the invariant bilinear form with dual basis

$$\bar{a}_{\mathbf{k},\mathbf{l}} = \prod_i \frac{1}{i^{r_i} r_i!} \prod_j \frac{1}{j^{s_j} s_j!} a_{\mathbf{l},\mathbf{k}}. \tag{24}$$

The basic Heisenberg 2-point function is

$$\langle \mathbf{1}, Y(a^+, x) Y(a^-, y)\mathbf{1}\rangle = \frac{1}{(x-y)^2}. \tag{25}$$

This function provides all the necessary data for computing the Heisenberg partition and correlation functions on a genus g surface! Thus the general rank 2 Heisenberg $2n$-point function is

$$\langle \mathbf{1}, Y(a^+, x_1) \ldots Y(a^+, x_n) Y(a^-, y_1) \ldots Y(a^-, y_n)\mathbf{1}\rangle = \mathrm{perm}\left(\frac{1}{(x_i - y_j)^2}\right). \tag{26}$$

This is a generating function for all rank two Heisenberg correlation functions by associativity of the VOA.

Let $x_{-i} = x - A_{-i}$ and $y_j = y - A_j$ be local coordinates in the neighborhood of canonical sewing parameters A_{-i}, A_j for $i, j \in \{\pm 1, \dots \pm g\}$ with $i \neq -j$. The 2-point function has expansion

$$\frac{1}{(x-y)^2} = \sum_{k,l \geq 1} (-1)^{k+1} \frac{(k+l-1)!}{(k-1)!(l-1)!} \frac{x_{-i}^{k-1} y_j^{l-1}}{(A_{-i} - A_j)^{k+l}}.$$

Define the canonical moment matrix R^{Can}, an infinite matrix indexed by $k, l = 1, 2, \dots$ and $i, j \in \{\pm 1, \dots \pm g\}$ where

$$R_{ij}^{\mathrm{Can}}(k,l) = \begin{cases} \frac{(-1)^k \rho_i^{k/2} \rho_j^{l/2}}{\sqrt{kl}} \frac{(k+l-1)!}{(k-1)!(l-1)!} \frac{1}{(A_{-i} - A_j)^{k+l}}, & i \neq -j \\ 0, & i = -j \end{cases} \tag{27}$$

$(I - R^{\mathrm{Can}})^{-1}$ plays a central role in computing the genus g period matrix and other structures.

We similarly have expansions in the Schottky parameters. Let

$$x_{-i} = \sigma_{-i}(x) = \frac{x - a_{-i}}{x - a_i} \tag{28}$$

$$y_j = \sigma_j(x) = \frac{y - a_j}{y - a_{-j}} \tag{29}$$

for $i, j \in \{1, \dots, g\}$ be local coordinates in the neighborhood of the Schottky points a_{-i} and a_j for $i \neq -j$. The 2-point function expansion leads to the Schottky moment matrix with

$$R_{ij}^{\mathrm{Sch}}(k,l) = \begin{cases} q_i^{k/2} q_j^{l/2} D(k,l)(\sigma_i \sigma_j^{-1}), & i \neq -j \\ 0, & i = -j \end{cases} \tag{30}$$

where for $\gamma \in SL(2, \mathbb{C})$

$$D(k,l)(\gamma) \;=\; \frac{1}{l!} \sqrt{\frac{l}{k}} \partial_z^l \left(\gamma(z)^k \right) |_{z=0}. \tag{31}$$

D is an $SL(2, \mathbb{C})$ representation [Mo]. Then it follows

$$\sum_{s \geq 1} R_{ij}^{\mathrm{Sch}}(r,s) R_{jk}^{\mathrm{Sch}}(s,t) = q_i^{r/2} q_k^{t/2} D(r,t)(\sigma_i \gamma_j \sigma_k^{-1}), \tag{32}$$

for Schottky generator (17).

4.3 The Genus g Partition Function - Canonical Parameters

We now define the genus g partition function for a VOA V in the canonical sewing scheme in terms of genus zero $2g$-point correlation functions as follows:

$$Z_V^{(g)}(\rho_i, A_{\pm i}) = \langle \mathbf{1}, \prod_{i=1}^{g} \sum_{n_i \geq 0} \rho_i^{n_i} \sum_{v_i \in V^{(n)}} Y(v_i, A_{-i}) Y(\overline{v}_i, A_i) \mathbf{1} \rangle, \tag{33}$$

where $\overline{v}_i$ is dual to v_i.

For genus one this reverts to the standard definition:

Theorem 4.2 (Mason and T.)

$$Z_V^{(1)}(\rho, A_{\pm 1}) = \mathrm{Tr}_V(q^{L_0})$$

where $q = C(\chi)$, *the Catalan series for* $\chi = -\frac{\rho}{(A_{-1} - A_1)^2}$.

4.4 $Z_{M_2}^{(g)}(\rho_i, A_{\pm i})$ for Heisenberg VOA M_2

The genus g partition function can be computed for the rank 2 Heisenberg VOA by means of the MacMahon Master Theorem where, schematically, we have:

Sum over g Fock bases	$\longrightarrow$	Sum over multisets
$2g$-point function	$\longrightarrow$	Permanent of matrix
Dual vector factorials	$\longrightarrow$	Multiset factorials
ρ_i and other dual vector factors	$\longrightarrow$	Absorbed into matrix definition

We then find that [TZ]

Theorem 4.3

$$Z_{M_2}^{(g)}(\rho_i, A_{\pm i}) = \frac{1}{\det(I - R^{\mathrm{Can}})},$$

where R^{Can} *is the canonical moment matrix. Furthermore,* $\det(I - R^{\mathrm{Can}})$ *is holomorphic and non-vanishing. In general, the genus g Heisenberg generating function is expressed in terms of a permanent of genus g bilinear forms of the second kind.*

We may repeat this by using an alternative definition of the genus g partition function in terms of in Schottky parameters account must be taken of the Möbius maps σ_i of (18). We then find [TZ]

Theorem 4.4 *The genus g partition function is*

$$Z_{M_2}^{(g)}(q_i, a_{\pm i}) = \frac{1}{\det(I - R^{\mathrm{Sch}})},$$

where R^{Sch} is the Schottky moment matrix. Furthermore, $\det(I - R^{\mathrm{Sch}})$ *is holomorphic and non-vanishing and the genus g Heisenberg generating function is expressed in terms of a permanent of genus g bilinear forms of the second kind.*

Conjecture: $\det(I - R^{\mathrm{Can}}) = \det(I - R^{\mathrm{Sch}})$. This is true for $g = 1$ [MT2].

4.5 The Montonen-Zograf Product Formula

$\det(I - R^{\mathrm{Sch}})$ can be also re-expressed in terms of an infinite product formula originally calculated in physics by Montonen in 1974 [Mo]. A similar product formula was subsequently found by Zograf [Z]. This has been recently related by McIntyre and Takhtajan [McT] to Mumford's theorem concerning the absence of a global section on moduli space for the canonical line bundle [Mu].

Recall that $R_{ij}^{\mathrm{Sch}}(k,l)$ is expressed in terms of an $SL(2,\mathbb{C})$ representation D. This leads to

$$\det(I - R^{\mathrm{Sch}}) = \prod_{m\geq 1} \prod_{\gamma^\alpha \in \Gamma} (1 - q_\alpha^m), \tag{34}$$

where the inner product ranges over the primitive elements γ^α of the Schottky group Γ i.e. $\gamma^\alpha \neq \gamma^k$ for any $\gamma \in \Gamma$ for $k > 1$. Each such element has a multiplier q_α where

$$\gamma^\alpha \sim \begin{pmatrix} q_\alpha & 0 \\ 0 & 1 \end{pmatrix}. \tag{35}$$

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